On the Status of Wormholes in Einstein's Theory: An Overview

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Abstract

It has been claimed that wormholes are just as good a prediction of Einstein's theory as black holes, but they are subject to severe restrictions from quantum field theory. The purpose of this paper is to show that the claim can be substantiated in spite of these restrictions.

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1. INTRODUCTION

The laws of physics are often used to make solid inferences. For example, Newton's laws allow the determination of the motion of a weight hanging on a spring. In other situations, the laws may simply allow something to happen. For example, Einstein's theory allows backward time travel but does not imply that backward time travel can actually be achieved. Similar comments can be made about macroscopic traversable wormholes: while wormholes are just as good a prediction of Einstein's theory as black holes, they are subject to severe restrictions from quantum field theory. An example is the need to violate the null energy condition, calling for the existence of "exotic matter" (defined below) to hold a wormhole open. Its problematical nature has caused many researchers to consider such wormhole solutions to be completely unphysical.

The continuing interest in wormholes is based on the observation that the Schwarzschild solution and therefore black holes describe a (nontraversable) wormhole. More recent developments involving entanglement have suggested that a special type of wormhole, called an Einstein-Rosen bridge, may be the best explanation for entanglement [1]. We will therefore assume that a basic wormhole structure can be hypothesized.

While there had been some forerunners, macroscopic traversable wormholes were first studied by Morris and Thorne [2], who proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad (1)$$

where

$$e^{2\alpha(r)} = \frac{1}{1 - \frac{b(r)}{r}}. \quad (2)$$

(We are using units in which $c = G = 1$.) In the now customary terminology, $\Phi = \Phi(r)$ is called the *redshift function*, which must be finite everywhere to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [2]. The spherical surface $r = r_0$ is the *throat* of the wormhole. In a Morris-Thorne wormhole, the shape function must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$, and $b'(r_0) < 1$, called the *flare-out condition* in [2]. In classical general relativity, the flare-out condition can only be met by violating the null energy condition (NEC), which states that for the energy-momentum tensor $T_{\alpha\beta}$

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0 \quad \text{for all null vectors } k^\alpha. \quad (3)$$

Matter that violates the NEC is called "exotic" in [2]. Applied to a wormhole setting, observe that for the radial outgoing null vector $(1, 1, 0, 0)$, the violation reads

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \quad (4)$$

Here, $T^t{}_t = -\rho(r)$ is the energy density, $T^r{}_r = p_r(r)$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t(r)$ is the lateral (transverse) pressure. Our final requirement is *asymptotic flatness*:

$$\lim_{r\to\infty}\Phi(r) = 0, \qquad \lim_{r\to\infty}\frac{b(r)}{r} = 0. \quad (5)$$

The problematical nature of exotic matter in conjunction with the need to violate the NEC has suggested solutions beyond the classical theory. For example, it was proposed by Lobo and Oliveira [3] that in $f(R)$ modified gravity, the wormhole throat could be lined with ordinary matter, while the violation of the NEC can be attributed to the higher-order curvature terms. There exist a number of other modified theories of gravity that could be called upon to address these issues.

The primary goal of this paper is to accommodate the energy violation without modifying Einstein's theory.

2. MEETING THE GOAL: PRELIMINARIES

When dealing with a complex theory such as general relativity, certain aspects can be viewed from a broader perspective that stops short of a modification. For example, the extension of Einstein's theory to higher dimensions has a long history, eventually leading to the realization that Einstein's theory is the low-energy limit of string theory (with its extra dimensions), just as Newton's theory is the weak-gravity and low-velocity limit of Einstein's theory. To be consistent with our goal, we will consider an extra dimension to be a natural extension of Einstein's theory, rather than a modification. So much of our interest is going to be centered on [4, 5], which hypothesize an extra static and time-dependent spatial dimension, respectively. These topics are covered in Sections 3–5.

Another striking development is noncommutative geometry, which may be viewed as another offshoot of string theory. As described in Section 6, point-like particles are replaced by smeared objects, which is consistent with Heisenberg's uncertainty principle. What is critically important for our purposes is that the noncommutative effects can be implemented in the Einstein field equations by modifying only the energy-momentum tensor while leaving the Einstein tensor intact, once again avoiding a modification of Einstein's theory. These topics are discussed in Sections 6 and 8.2.

The realization that moderately sized wormholes are subject to an enormous radial tension suggests that wormholes are actually compact stellar objects. A possible explanation is sought in Section 7 by starting with a two-fluid model that was previously proposed in [18]. Additional assumptions are unavoidable, however, as we will see in Section 7.

Exotic matter makes a brief comeback in Section 8. Small amounts that may arise from, for example, the Casimir effect, call for striking a delicate balance between reducing the amount of exotic matter and fine-tuning the metric coefficients.

Finally, it is shown in Section 9 that a noncommutative-geometry wormhole in a static and spherically symmetric spacetime admitting conformal motion is stable to linearized radial perturbations. Furthermore, both the redshift and shape functions are completely determined from the given conditions.

In Section 10, we take another, more general, look at the low energy density in a noncommutative-geometry setting by comparing the outcome to other low-density models, including the case $\rho(r) \equiv 0$; none of these have the special characteristics of the former. Without these special features, the need for exotic matter cannot be avoided, indicating that neither dark matter nor dark energy can support traversable wormholes, at least not as long as the latter does not cross the phantom divide.

Section 11 discusses the possible detection of wormholes by means of gravitational lensing. This tool calls for additional physical requirements beyond the existence of dark matter, thereby confirming the above assertion.

3. AN EXTRA SPATIAL DIMENSION (STATIC CASE)

In this section, our main interest will be centered on [4], which involves an extra spatial dimension. The extended line element is

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{2\mu(r,l)}dl^2, \tag{6}$$

where l is the extra coordinate.

It is interesting to note that in the components of the Riemann curvature tensor, $\mu(r,l)$ never occurs as a factor. Instead, the factors are $\partial\mu(r,l)/\partial r$ and $\partial^2\mu(r,l)/\partial r^2$ [4]. So, $\mu(r,l)$ could have any magnitude, to be discussed further below.

According to [4],

$$(\rho + p_r)|_{r=r_0} = \frac{1}{8\pi}\frac{b'(r_0)-1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial\mu(r_0,l)}{\partial r}\right]. \tag{7}$$

To satisfy the condition $\rho + p_r > 0$ at the throat, we must have

$$\frac{\partial\mu(r_0,l)}{\partial r} < -\frac{2}{r_0}, \tag{8}$$

corresponding to the null vector $(1,1,0,0)$. Moving to the fifth dimension, the null vector $(1,0,0,0,1)$ yields

$$(\rho + p_r)|_{r=r_0} = \frac{1}{8\pi}\frac{1}{2}\frac{rb'-b}{r^2}\left[-\frac{d\Phi(r)}{dr} + \frac{\partial\mu(r,l)}{\partial r}\right]\Bigg|_{r=r_0} < 0, \tag{9}$$

provided that the redshift function satisfies a similar condition:

$$\frac{d\Phi(r_0)}{dr} = -A < \frac{\partial\mu(r_0,l)}{\partial r} < -\frac{2}{r_0}. \tag{10}$$

We conclude that the NEC is satisfied at the throat in the four-dimensional spacetime but violated in the five-dimensional spacetime.

Remark 1. For the condition $\rho(r_0) + p_r(r_0) > 0$ to hold for all null vectors, we must also have $b'(r_0) > 1/3$ [4].

Remark 2. Condition (10) can be readily satisfied if $\Phi = \Phi(r)$ is a positive differentiable decreasing function of r for all r. The reason is that since $\Phi'(r) < 0$, the assumption of asymptotic flatness implies that $\lim_{r\to\infty}\Phi'(r) = 0$.

4. ADDITIONAL CONSIDERATIONS

4.1. The Function $\mu = \mu(r,l)$

We have already seen that inequality (8) is a sufficient condition for ensuring that the throat of a wormhole can be threaded with ordinary matter, while the unavoidable violation of the NEC can be attributed to the higher spatial dimension.

Here, we need to emphasize another aspect of $\mu(r,l)$: as noted in the previous section, $|\mu(r,l)|$ can be large or small. So, in our model, it is entirely possible that μ be negative with a large absolute value, resulting in a small value for $e^{\mu(r,l)}$. In other words, the extra dimension could be compactified without sacrificing inequality (8). The very existence of a compactified extra dimension is consistent with string theory. So, the assumptions regarding $\mu(r,l)$ are physically reasonable.

Summarizing the static case, $\Phi = \Phi(r)$ is positive and decreasing, while $b'(r_0) > 1/3$. Conditions (8) and (10) are physically reasonable and consistent with string theory.

4.2. The Radial Tension at the Throat

At this point, we need to return to [2] to discuss the radial tension at the throat. To that end, we need to recall that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$. According to [2], the Einstein field equations can be rearranged to yield $\tau(r)$. Here, we need to reintroduce c and G temporarily to get

$$\tau(r) = \frac{b(r)/r - 2[r-b(r)]\Phi'(r)}{8\pi G c^{-4} r^2}. \tag{11}$$

So, the radial tension at the throat becomes

$$\tau(r_0) = \frac{1}{8\pi G c^{-4} r_0^2} \approx 5\times 10^{41}\,\frac{\text{dyn}}{\text{cm}^2}\left(\frac{10\,\text{m}}{r_0}\right)^2. \tag{12}$$

As pointed out in [2], for $r_0 = 3\,\text{km}$, $\tau(r)$ has the same magnitude as the pressure at the center of a massive neutron star. (For further discussion of this problem, see [7].) So, it follows from equation (12) that wormholes with a low radial tension could only exist on very large scales.

5. AN EXTRA SPATIAL DIMENSION (TIME-DEPENDENT CASE)

So far, the wormhole geometry has been strictly static. Reference [5] discusses the case in which the extra dimension is a function of time t, as well as r and l. If the shape and redshift functions remain the same, then the line element becomes

$$\begin{aligned} ds^2 = &-e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} \\ &+ r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{2\mu(r,l,t)}dl^2. \end{aligned} \tag{13}$$

(Reference [5] also assumes that b and Φ are functions of r and l.) The expression for the null energy condition is given by

$$\begin{aligned} 8\pi\,(\rho + p_r)|_{r=r_0} = &\frac{b'(r_0)-1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial\mu(r_0,l,t)}{\partial r}\right] \\ &- e^{-2\Phi(r_0)}\left[\frac{\partial^2\mu(r_0,l,t)}{\partial t^2} + \left(\frac{\partial\mu(r_0,l,t)}{\partial t}\right)^2\right]. \end{aligned} \tag{14}$$

To put this result to use, we can start with [8], which deals with a wormhole model due to S.-W. Kim [9] in conjunction with a generalized Kaluza-Klein model:

$$\begin{aligned} ds^2 = &-e^{2\Phi(r)}dt^2 + [a(t)]^2 \\ &\times\left(\frac{dr^2}{1-kr^2-\frac{b(r)}{r}} + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right) + e^{2\Psi(r)}dq^2\right). \end{aligned} \tag{15}$$

However, our primary concern is the effect of the time-dependent extra dimension, rather than the overall cosmological model. So, we can let $k = 0$ and assume that the line element has the form

$$\begin{aligned} ds^2 = &-e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} \\ &+ r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + [a(t)]^2 e^{2\mu(r,l)}dl^2, \end{aligned} \tag{16}$$

using our earlier notation for the last term. Since $[a(t)]^2e^{2\mu(r,l)} = e^{2(\ln a(t)+\mu(r,l))}$, we let

$$U = \ln a(t) + \mu(r,l). \tag{17}$$

Then, the line element becomes

$$\begin{aligned} ds^2 = &-e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} \\ &+ r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{U(r,l,t)}dl^2. \end{aligned} \tag{18}$$

So, we can now rewrite equation (14) in the form

$$\begin{aligned} &8\pi\,(\rho + p_r)|_{r=r_0} \\ &= \frac{b'(r_0)-1}{r_0^2}\left[1 + \frac{r_0}{2}\frac{\partial U(r_0,l,1)}{\partial r}\right] \\ &- e^{-2\Phi(r_0)}\left[\frac{\partial^2 U(r_0,l,t)}{\partial t^2} + \left(\frac{\partial U(r_0,l,t)}{\partial t}\right)^2\right]. \end{aligned} \tag{19}$$

We then observe that

$$\frac{\partial^2 U}{\partial t^2} + \left(\frac{\partial U}{\partial t}\right)^2 = \frac{a''(t)}{a(t)}. \tag{20}$$

There are now two possibilities, $a''(t) < 0$ and $a''(t) > 0$ for the nonstatic case. In the first case, the second term on the right-hand side of equation (19) is positive. It now becomes apparent that inequality (8) can be replaced by the slightly more general

$$\frac{\partial\mu(r_0,l,t)}{\partial r} \le -\frac{2}{r_0} \tag{21}$$

since

$$\frac{\partial U}{\partial r} = \frac{\partial\mu}{\partial r}.$$

In other words, from equation (19), we now have

$$8\pi\,(\rho + p_r)|_{r=r_0} > 0, \tag{22}$$

as in the static case.

If $a''(t) > 0$, then

$$8\pi\,(\rho + p_r)|_{r=r_0} < 0, \tag{23}$$

and we are back to the exotic matter, the case that was previously dismissed as unphysical.

Remark 3. It is shown in [5] that the NEC is violated in the five-dimensionsl spacetime, as in the static case.

6. MACROSCOPIC WORMHOLES AS EMERGENT PHENOMENA

In this section, our study of wormholes will move in a different direction by starting with noncommutative geometry. Sometimes viewed as an offshoot of string theory, it assumes that point-like particles are replaced by smeared objects, which is consistent with the Heisenberg uncertainty principle. The original idea was to eliminate the divergences that normally occur in general relativity [10, 11, 12]. According to [11], this objective can be met by asserting that spacetime can be encoded in the commutator $[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck's constant $\hbar$ discretizes phase space. More concretely, the smearing can be modeled by using a so-called Lorentzian distribution of minimal length $\sqrt{\beta}$ instead of the Dirac delta function [13, 14]. As a consequence, the energy density of a static and spherically symmetric and particle-like gravitational source is given by

$$\rho(r) = \frac{m\sqrt{\beta}}{\pi^2\left(r^2+\beta\right)^2}. \tag{24}$$

The implication is that the gravitational source causes the mass m to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty.

Returning to line element (1), let us list the Einstein field equations next:

$$\rho(r) = \frac{b'}{8\pi r^2}, \tag{25}$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{26}$$

$$p_t(r) = \frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right]. \tag{27}$$

Equation (24) now provides a physical basis for checking the NEC, i.e.,

$$\begin{aligned} T_{\alpha\beta}k^\alpha k^\beta = \rho(r) + p_r(r) &= \frac{m\sqrt{\beta}}{\pi^2\left(r^2+\beta\right)^2} + \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right]\Bigg|_{r=r_0} \\ &= \frac{m\sqrt{\beta}}{\pi^2\left(r_0^2+\beta\right)^2} - \frac{1}{8\pi}\frac{b(r_0)}{r_0^3} < 0 \end{aligned} \tag{28}$$

since $\sqrt{\beta} \ll m$. So, the violation of the NEC can be attributed to the noncommutative-geometry background, rather than some hypothetical "exotic matter," *at least locally*. (We will return to this point at the end of the section.)

For our purposes, it is sufficient to note that, according to [6], the shape function B is given by

$$\begin{aligned} B\left(\frac{r}{\sqrt{\beta}}\right) &= \frac{4m}{\pi}\frac{1}{r}\left[\frac{r}{\sqrt{\beta}}\tan^{-1}\frac{r}{\sqrt{\beta}} - \frac{\left(\frac{r}{\sqrt{\beta}}\right)^2}{\left(\frac{r}{\sqrt{\beta}}\right)^2+1}\right. \\ &\left. - \frac{r}{\sqrt{\beta}}\tan^{-1}\frac{r_0}{\sqrt{\beta}} + \frac{r}{\sqrt{\beta}}\frac{\frac{r_0}{\sqrt{\beta}}}{\left(\frac{r_0}{\sqrt{\beta}}\right)^2+1}\right] + \frac{r_0}{\sqrt{\beta}} \end{aligned} \tag{29}$$

and meets all the requirements of a shape function. In particular,

$$B\left(\frac{r_0}{\sqrt{\beta}}\right) = \frac{r_0}{\sqrt{\beta}}, \tag{30}$$

which corresponds to $b(r_0) = r_0$. It follows that the throat radius is macroscopic. (See [6] for details.)

This outcome naturally raises the question whether a modification of Einstein's theory has really been avoided. It is argued in [6] that noncommutative geometry in the form discussed above is a *fundamental property* and that the outcome, a macroscopic throat size, is an *emergent property*. By definition, emergent phenomena are derived from some fundamental theory, an idea that dates at least from the time of Aristotle. For example, life emerges from totally lifeless objects, such as atoms and molecules. This process is not reversible: living organisms tell us little about the particles in the fundamental theory. Similarly, our emerging macroscopic scale does not yield the smearing effect in the fundamental theory. In the usual terminology, we have obtained an *effective model* for a macroscopic wormhole in the sense that the short-distance effects have been discarded: these are meaningful only in the fundamental theory. The above local violation of the NEC, $T_{\alpha\beta}k^\alpha k^\beta < 0$, can therefore be viewed as a fundamental property. So, the emergent macroscopic phenomenon in equation (30) avoids a modification of Einstein's theory.

7. NEUTRON STARS

With Section 4.2 in mind, wormholes should be viewed as compact stellar objects. The reason is that $\tau(r)$ has the same magnitude as the pressure at the center of a massive neutron star. Moreover, equation (12) implies that a wormhole with a low radial tension could only exist on a very large scale, i.e., with a sufficiently large $r = r_0$. According to [15], for smaller wormholes, even the boundary condition $b(r_0) = r_0$ only makes sense if the wormhole is a compact stellar object.

It is interesting to note that a combined model consisting of neutron-star matter and a phantom/ghost scalar field yields a wormhole solution [16]. Another example of a two-fluid model can be found in [18]. For this approach to work, we need to follow [17] which assumes that quark matter exists at the center of neutron stars. While this may seem like a strong assumption, it is by no means unreasonable: the extreme conditions could presumably cause the neutrons to become deconfined, resulting in quark matter. Armed with this assumption, the energy-momentum tensor of the two-fluid model is given by [18]

$$T_0^0 \equiv \rho_{\text{effective}} = \rho + \rho_q, \tag{31}$$

$$T_1^1 = T_2^2 \equiv -p_{\text{effective}} = -\left(p + p_q\right). \tag{32}$$

Here, ρ and p correspond to the respective energy density and pressure of the baryonic matter, while ρ_q and p_q correspond to the respective energy density and pressure of the quark matter. The left-hand sides are the effective energy density and pressure, respectively, of the combination.

The two-fluid model is based on the MIT bag model [19]. In this model, the equation of state is given by

$$p_q = \frac{1}{3}\left(\rho_q - 4B\right), \tag{33}$$

where B is the bag constant, which is given as $145\,\text{MeV}/(\text{fm})^3$ in [19]. For normal matter, we can use the rather idealized equation of state [20]

$$p = m\rho, \quad 0 < m < 1. \tag{34}$$

For our purposes, it is sufficient to note that [17] gives the following solution:

$$\rho = \rho_0 e^{-\Phi(1+m)/2m}, \tag{35}$$

$$\rho_q = B + \rho_{(q,0)}e^{-2\Phi}, \tag{36}$$

where ρ_0 and $\rho_{(q,0)}$ are integration constants. Reference [17] then goes on to derive the shape function $b = b(r)$, as well as its derivative

$$b'(r) = 1 - e^{-\alpha(r)} + r\left[-\frac{d}{dr}e^{-\alpha(r)}\right], \tag{37}$$

where $e^{\alpha(r)} = 1/(1 - b(r)/r)$. (See [17] for details.) It is subsequently shown that the flare-out condition is met, indicating a violation of the null energy condition, a necessary condition for the existence of wormholes. This violation can be attributed to the extreme conditions at the center of neutron stars.

8. EXOTIC MATTER REVISITED

8.1. Fine-Tuning

Given our main goal, demonstrating the possible existence of wormholes without modifying Einstein's theory, it seems surprising that an earlier attempt by the author required no more than sufficient fine-tuning of the metric coefficients [21].

First, we need to recall that in classical general relativity, a wormhole can only be held open by violating the NEC, calling for the need for exotic matter. Such matter is confined to a small region around the throat. By itself, this is not a conceptual problem, as shown by the Casimir effect [22], to be discussed further in the next section. In other words, exotic matter can be made in the laboratory, but only in small quantities that may not be sufficient for keeping a wormhole open. One of the goals in [21] is to strike a balance between two conflicting requirements, reducing the amount of exotic matter and fine-tuning the values of the metric coefficients.

The key to the problem is the discovery by Ford and Roman that quantum field theory places severe constraints on the wormhole geometries [23, 24]. Of particular interest to us is equation (95) in [24]:

$$\frac{r_m}{r} \leq \left(\frac{1}{v^2 - b'(r_0)}\right)^{1/4} \frac{\sqrt{\delta}}{f} \left(\frac{l_p}{r_0}\right)^{1/2}, \tag{38}$$

where $\delta = 1/\sqrt{1-v^2}$, l_p is the Planck length, f is a small-scale factor, $b'_0 = b'(r_0)$, and r_m is the smallest of several length scales:

$$r_m \equiv \min\left[b(r), \left|\frac{b(r)}{b'(r)}\right|, \frac{1}{|\Phi'(r)|}, \left|\frac{\Phi'(r)}{\Phi''(r)}\right|\right], \tag{39}$$

referring once again to line element (1). Finally, v is the velocity of a boosted observer relative to a static frame. For the right-hand side of inequality (38) to be defined and real, we must have $v^2 > b'_0$. So, if $b'_0 \approx 1$, the inequality is trivially satisfied, thereby meeting the Ford-Roman constraints. However, to study the region away from the throat, where $b'(r_0) < 1$, inequality (38) has to be extended, as we will see shortly.

Before continuing, we need to take a closer look at the exotic region

$$l(r) = \int_{r_0}^{r} e^{\alpha(r')} dr'. \tag{40}$$

So, $l(r_1)$ is the amount of exotic matter in the interval $[r_0, r_1]$. (This is a more precise way of saying that the exotic matter is confined to the spherical shell of inner radius $r = r_0$ and outer radius $r = r_1$.)

Now, consider the extended quatum inequality from [21]:

$$\frac{r_m}{r} \leq \left(\frac{1}{v^2 \frac{b(r)}{r} - b'(r) - 2v^2\Phi'(r)\left(1 - \frac{b(r)}{r}\right)}\right)^{1/4} \frac{\sqrt{\delta}}{f}\left(\frac{l_p}{r}\right)^{1/2}. \tag{41}$$

At $r = r_0$, inequality (41) reduces to inequality (38). As a result, we are still interested in the case where $b'(r_0)$ is close to unity because this leads to our main result: since $b'(r) < 1$, for $r > r_0$, inequality (38) is not necessarily satisfied, but in the extended inequality (41), $\Phi'(r)$ can be fine-tuned so that the condition is satisfied in the interval $[r_0, r_1]$, thereby reducing the proper thickness of the exotic region, perhaps indefinitely. (Reference [21] discusses additional models and gives several numerical estimates.)

The conclusion is that one must strike a balance between the thickness $[r_0, r_1]$ of the exotic region and the degree of fine-tuning required to achieve this reduction. It is also shown in [21] that the degree of fine-tuning is a generic feature of a Morris-Thorne wormhole. This unexpected finding could be viewed as an engineering challenge that some day might even be met.

8.2. The Casimir Effect and Noncommutative Geometry

The Casimir effect mentioned in Section 8.1 has shown that exotic matter can exist on a small scale. While this may not be enough to guarantee that the Casimir effect can support a macroscopic wormhole, the fine-tuning scheme in Section 8.1 seems to allow such a possibility. Another possibility is discussed in [22]: the Casimir effect can be connected to noncommutative geometry, which also deals with small-scale effects, as discussed in Section 6.

The Casimir effect is usually described by starting with two closely spaced parallel metallic plates in a vacuum. These can be replaced by two closely spaced concentric spheres to preserve the spherical symmetry. According to [25], if a is the magnitude of the separation, then the pressure p as a function of a is given by

$$p(a) = -3\frac{\hbar c \pi^2}{720 a^4} \tag{42}$$

and the density is

$$\rho_C(a) = -\frac{\hbar c \pi^2}{720 a^4}. \tag{43}$$

Here, $\hbar$ is Planck's constant and c is the speed of light.

At this point we are going to return to noncommutative geometry by recalling the form of the energy density in equation (24) and its interpretation: the gravitational source causes the mass m of a particle to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty. Here, we are going to be more concerned with a smeared spherical surface of which the throat of a wormhole is our primary example. According to [22], the energy density ρ_s is given by

$$\rho_s(r - r_0) = \frac{\mu\sqrt{\beta}}{\pi^2\left[(r - r_0)^2 + \beta\right]^2}, \tag{44}$$

where μ now denotes the mass of the surface. So, the smeared particle is replaced by a smeared surface.

To connect the Casimir effect to the noncommutative-geometry background, we first observe from equation (24) that the energy density ρ as a function of the separation a is

$$\rho(a) = \frac{m\sqrt{\beta}}{\pi^2 (a^2 + \beta)^2}. \tag{45}$$

According to equation (44), in the vicinity of the throat, i.e., whenever $r - r_0 = a$, we get

$$\rho_s(a) = \frac{\mu\sqrt{\beta}}{\pi^2 (a^2 + \beta)^2}. \tag{46}$$

So, it follows from equation (43) that

$$\frac{\mu\sqrt{\beta}}{\pi^2\left(a^2+\beta\right)^2} = |\rho_C(a)| = \frac{\hbar c\pi^2}{720a^4}, \tag{47}$$

which is the sought-after connection. More precisely, it is argued in [22] that the separation a, although small, is still macroscopic. So, we can assume that $\beta = (\sqrt{\beta})^2 \ll a^2$. Moreover, since β is an additive constant, it becomes negligible in the denominator of equation (47); thus,

$$\sqrt{\beta} = \frac{\hbar c\pi^4}{720\mu}. \tag{48}$$

Since $\hbar = 1.0546 \times 10^{-34}\,\mathrm{J}\cdot\mathrm{s}$, we obtain

$$\sqrt{\beta} = \frac{4.28 \times 10^{-27}}{\mu}. \tag{49}$$

Given that μ is the mass of the throat $r = r_0$, a spherical surface of negligible thickness, it is hard to quantify, but it does have a definite value, thereby defining $\sqrt{\beta}$ in equation (49).

It is proposed in [22] that we could give a direct physical interpretation to the smearing effect by letting $\sqrt{\beta} = a$. Then, equation (47) yields

$$\frac{\mu a}{\pi^2\left(a^2+a^2\right)^2} = \frac{\hbar c\pi^2}{720a^4} \tag{50}$$

or

$$\mu a = \frac{\hbar c\pi^4}{180}, \tag{51}$$

a fixed quantity. So, there are many possible choices for a and μ.

Successfully connecting the experimentally confirmed Casimir effect to noncommutative geometry has some important consequences. Here, we can follow the arguments proposed in [11], starting with the assertion that the noncommutative effects can be implemented in the Einstein field equations $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$ by modifying only the energy-momentum tensor $T_{\mu\nu}$, while leaving the Einstein tensor $G_{\mu\nu}$ intact. The reason given in [11] is that a metric field is a geometric structure defined over an underlying manifold whose strength is measured by its curvature, but the curvature, in turn, is nothing more than the response to the presence of a mass-energy distribution. Moreover, the noncommutativity is an intrinsic property of spacetime, rather than a superimposed geometric structure. So, it stands to reason that noncommutative geometry has an effect on the mass-energy and momentum distributions, which, in turn, determines the spacetime curvature. None of this affects the Einstein tensor. So, the length scales can be macroscopic. (Recall that we already saw in Section 6 that the throat radius can be macroscopic.)

In summary, by invoking noncommutative geometry, we have seen that the Casimir effect, although a small effect, may very well support a macroscopic wormhole.

9. STABILITY

The possible existence of macroscopic traversable wormholes has naturally led to numerous studies regarding the stability of such structures. We are going to confine ourselves to [26] because the assumption of conformal symmetry in [26] can be combined with the noncommutative-geometry background to produce a complete wormhole solution. It is assumed in [26] that our static and spherically symmetric spacetime admits a one-parameter group of conformal motions. This assumption is equivalent to the existence of conformal Killing vectors such that

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\eta\nu}\,\xi^\eta{}_{;\mu} + g_{\mu\eta}\,\xi^\eta{}_{;\nu} = \psi(r)\,g_{\mu\nu}, \tag{52}$$

where the left-hand side is the Lie derivative of the metric tensor and $\psi(r)$ is the conformal factor [26]. According to the usual terminology, ξ generates the conformal symmetry and the metric tensor $g_{\mu\nu}$ is said to be conformally mapped into itself along ξ. This type of symmetry has been used extensively in classical general relativity.

Before returning to the stability question, we need to recall the usual strategy in the theoretical construction of a Morris-Thorne wormhole: retain complete control over the geometry by specifying the redshift and shape functions and then manufacture or search the Universe for materials or fields that produce the required stress-energy tensor. Reference [27] addresses this problem in a direct manner: the noncommutative-geometry background produces the shape function and the conformal symmetry yields the redshift function. Adding the assumption of the conservation of mass energy then yields the stress-energy tensor. The result is a complete wormhole solution determined from the given conditions. Finally, it is shown that the wormhole is stable to linearized radial perturbations.

10. OTHER LOW-ENERGY-DENSITY WORMHOLES

Returning to equation (28), we have seen that the local violation of the NEC can be viewed as a fundamental property in a noncommutative-geometry setting from which emerges the violation on a macroscopic scale. This observation is consistent with equation (25), restated here for convenience:

$$\frac{b'(r)}{8\pi r^2} = \rho(r); \tag{53}$$

the left-hand side is the G_{tt} component of the Einstein tensor, which, as we saw in Section 8.2, is unaffected in a noncommutative-geometry setting. So, we are justified in using $\rho(r)$ from equation (24) in equation (53). It is interesting to note that the small value of $\rho(r)$ typically leads to $b'(r) < 1$. So, the flare-out condition is met automatically.

The real question now becomes, what if $\rho(r)$ has a small value but is otherwise arbitrary? While we still have $b'(r) < 1$, without the special features from the noncommutative-geometry background, we cannot simply and uncritically draw the same conclusions. To see why, consider an extreme example, the zero-density case $\rho \equiv 0$, treated in Visser's book [28]. We get a valid wormhole solution only if we go back to requiring the usual exotic matter. Unfortunately, similar comments can be made about various dark-matter models all of which have a very low energy density. Consider, for example, the Navarro-Frenk-White model in [29]:

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2}, \tag{54}$$

where r_s is the characteristic scale radius and ρ_s the corresponding density. Since $\rho(r)$ in equation (54) is very small, we would normally satisfy the flare-out condition $b'(r) < 1$ for all r, thereby yielding a wormhole solution. However, without the noncommutative-geometry background and its many special properties, we can no longer assume the validity of equation (53) for an arbitrary $\rho(r)$, unless, of course, we return to the exotic matter requirement once again. While this outcome does not invalidate the solutions, it does call into question their relevance: if exotic matter is needed anyway, then what is the role of dark matter, if any? In other words, if exotic matter cannot be eliminated, then dark matter alone could not support traversable wormholes. The same comments apply equally well to dark-energy models that do not cross the phantom divide. To clarify this point, we need to recall that for phantom dark energy, the (isotropic) equation of state is $p = \omega\rho$, $\omega < -1$, which implies that $\rho + p = \rho + \omega\rho = \rho(1+\omega) < 0$. Since the NEC has been violated, phantom dark energy could in principle support traversable wormholes [30]. Such wormholes could only exist on very large scales, however, as we already noted in Section 4.2.

11. A SOLUTION UNCOVERED VIA GRAVITATIONAL LENSING

We know from the previous section that neither dark matter alone nor dark energy alone can support a Morris-Thorne wormhole: the former requires the existence of exotic matter and the latter the equation of state $p = \omega\rho$, $\omega < -1$. Another possibility is a noncommutative-geometry background, as we saw in Section 6. This section considers yet another approach, discussing the effect of gravitational lensing. While primarily a tool for detecting wormholes, it has its own physical requirements, as described in [31]. To facilitate the discussion, the line element is written in the following more convenient form:

$$ds^2 = -A(x)\,dt^2 + B(x)\,dx^2 + C(x)\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \quad (55)$$

where x is the radial distance defined in terms of the Schwarzschild radius $x = r/2M$. Then,

$$x_0 = \frac{r_0}{2M} \quad (56)$$

denotes the closest approach of the light ray. As noted in [31], the deflection angle $\alpha(r_0)$ is given by

$$\alpha(r_0) = I(x_0) + a, \quad (57)$$

where a is a constant that depends on the size of the wormhole. Next,

$$\begin{aligned} I(x_0) &= 2\int_{x_0}^{\infty} \frac{\sqrt{B(x)}\,dx}{\sqrt{C(x)}\sqrt{\frac{C(x)A(x_0)}{C(x_0)A(x)} - 1}} \\ &= \int_{x_0}^{a} Q(x)\,dx. \end{aligned} \quad (58)$$

Here, $Q(x)$ depends on the parameters in the Navarro-Frenk-White model, equation (54). (See [31] for details.)

So, while we are still dealing with dark matter, the wormhole solution requires several other conditions besides the simple existence of dark matter as previously claimed. In particular, the deflection angle depends on both the redshift and shape functions.

12. CONCLUSION

Given that wormholes are just as good a prediction of Einstein's theory as black holes, we can assume that Morris-Thorne wormholes, as proposed in [2], are theoretically possible, but subject to severe restrictions from quantum field theory. The purpose of this paper is to show that these restrictions can be met without a modification of Einstein's theory.

Adhering to the widely held view that the need for exotic matter renders any wormhole solution unphysical, we follow [6] which proposes the following static and spherically symmetric line element to describe a wormhole spacetime:

$$\begin{aligned} ds^2 = &-e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} \\ &+ r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) + e^{2\mu(r,l)}dl^2, \end{aligned}$$

where l is the extra coordinate. The extra dimension can be either static or time dependent. For this approach to work, we do need to make some additional assumptions. For the static case, the redshift function is positive and decreasing for all r. (Asymptotic flatness is part of the structure of a Morris-Thorne wormhole.) Furthermore, inequalities (8) and (10) have to be met, while the condition $b'(r_0) > \frac{1}{3}$ ensures that the NEC is satisfied at the throat for all null vectors. So, the throat of the wormhole can be lined with ordinary matter, while the unavoidable violation of the NEC can be attributed to the higher spatial dimension. Finally, the extra dimension can be small or even curled up.

For the time-dependent case, we obtain a similar conclusion using the slightly more general condition $\partial\mu(r_0, l, t)/\partial r \leq -2/r_0$, provided that $a''(t) < 0$.

The next part of this paper invokes a noncommutative-geometry background, thereby assuming that point particles are replaced by smeared objects, as detailed in Section 6. This assumption is consistent with the Heisenberg uncertainty principle and therefore independent of Einstein's theory. The particle-like gravitational source is given by

$$\rho(r) = \frac{m\sqrt{\beta}}{\pi^2\left(r^2 + \beta\right)^2}. \quad (59)$$

So, the gravitational source causes the mass m of the particle to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty.

The shape function B meets all the usual requirements; in particular,

$$B\left(\frac{r_0}{\sqrt{\beta}}\right) = \frac{r_0}{\sqrt{\beta}}. \quad (60)$$

The throat radius $r_0/\sqrt{\beta}$ is therefore macroscopic.

It is argued in Section 6 that the noncommutative-geometry background is a *fundamental property* and the outcome, a macroscopic wormhole, is an *emergent phenomenon*. The result is an *effective model* that does not depend on the short-distance effect that is characteristic of noncommutative geometry, thereby avoiding a modification of Einstein's theory. It is interesting to note that the compactified extra spatial dimension in Section 4.1 can also be viewed as a fundamental property, again making the macroscopic wormhole an emergent phenomenon.

It was pointed out in Section 4.2 that for a wormhole with a throat radius of 3 km, the radial tension has the same magnitude as the pressure at the center of a massive neutron star, suggesting that a Morris-Thorne wormhole should be viewed as a compact stellar object. This case is taken up in Section 7 by first noting that quark matter is believed to exist near the center of neutron stars, thereby calling for a combined model consisting of quark matter and ordinary matter. For this type of wormhole, the violation of the null energy condition can be attributed to the extreme conditions at the center of the neutron star.

In Section 8.1, we saw a return to exotic matter, motivated in part by the fact that small amounts of exotic matter can be made in the laboratory, as exemplified by the Casimir effect. To get a valid wormhole solution, the wormhole has to satisfy the Ford-Roman inequality (38) or the extended version, inequality (41). The conclusion is that one must strike a balance between the thickness $[r_0, r_1]$ of the exotic region and the degree of fine-tuning required to achieve this reduction. The degree of fine-tuning is a generic feature of a Morris-Thorne wormhole. Section 8.2 then connects the aforementioned Casimir effect with noncommutative geometry, suggesting that the former may be able to support a macroscopic wormhole in spite of being a small effect.

Finally, it is shown in Section 9 that given a noncommutative-geometry background, a Morris-Thorne wormhole in a static and spherically symmetric spacetime admitting conformal motion is stable to linearized radial perturbations. Furthermore, the redshift and shape functions are completely determined from the given conditions.

In Section 10, we return to equation (53) to observe that $b'(r) = 8\pi r^2 \rho(r) < 1$ whenever $\rho(r)$ is extremely small, which is actually true in a dark-matter or dark-energy setting, as well as for the zero-density case $\rho(r) \equiv 0$. So, the NEC is automatically violated. The same is true for $\rho(r)$ in equation (24), the noncommutative-geometry case. The difference is that the use of $\rho(r)$ in equation (24) can be justified by appealing to the special properties of the noncommutative-geometry background, thereby producing a valid wormhole solution. Since these key properties are not possessed by any of the other cases, we conclude that neither dark matter nor dark energy can support a Morris-Thorne wormhole, as long as the latter does not cross the phantom divide. Other possible exceptions are noted in [31].

Section 11 discusses the detection of wormholes by means of gravitational lensing. An application of the method calls for additional physical requirements beyond the simple existence of dark matter, confirming the earlier assertion that dark matter alone cannot support traversable wormholes.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this paper.

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